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ON THE STRUCTURE AND THE HAUSDORFF DIMENSION OF THE
SUPPORT OF A CLASS OF DISTRIBUTION FUNCTIONS
INDUCED BY ERGODIC SEQUENCES

TECHNICAL REPORT NO. ESD-TR-65-130

OCTOBER 1965

H. Dym

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AIR FORCE SYSTEMS COMMAND

UNITED STATES AIR FORCE

L. G. Hanscom Field, Bedford, Massachusetts



Project 508G

Prepared by

THE MITRE CORPORATION

Bedford, Massachusetts

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ABSTRACT

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REVIEW AND APPROVAL

This technical report has been reviewed and is approved.



CHARLES A LAUSTRUP

Lt Col, USAF

Acting Director of Computers

Deputy for Engineering and Technology

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SECTION I

INTRODUCTION

Let X_1, X_2, \dots be a stationary ergodic sequence of random variables defined on a probability space (Ω, A, P) and mapping into the set $\{0, 1, \dots, D-1\}$. In particular, we may envision Ω as the set of all sequences $\omega = (\omega_1, \omega_2, \dots)$ with $X_n(\omega) = \omega_n$ and $\omega_n \in \{0, 1, \dots, D-1\}$. To avoid trivialities we assume that $D \geq 2$ and $P(\omega: X_1(\omega)=i) > 0$ for $i = 0, 1, \dots, D-1$.

The random variable $Z = \sum_{i=1}^{\infty} X_i D^{-i}$ may be used to transfer the probability structure of the random sequence to the unit interval; each point $\omega \in \Omega$ is mapped by Z into a point $Z(\omega)$ satisfying $0 \leq Z(\omega) \leq 1$. In this note we study the distribution function of Z , $F(a) = P(\omega: Z(\omega) < a)$. We show (Theorem 1) that either: F is a step function with k jumps of height $1/k$, or F is continuous and purely singular, or else $F(a) = a$; $F(a) = a$ if and only if the random variables X_i are independent and uniformly distributed. Harris [5] has proved an analogous theorem using different techniques. Our method of proof utilizes Information Theory and illustrates some interesting relationships between the entropy rate, H , of the random sequence and the form of the distribution function. In particular, H is maximal; that is $H = \log D = 1$ (all logarithms will be taken to the base D), if and only if $F(a) = a$. If $0 < H < 1$ then F is purely singular and continuous. If F is a step function then $H = 0$. Presumably there exist ergodic sequences for which $H = 0$ and F is continuous and purely singular though we have not been able to construct any.

In the proof of Theorem 1 we define a set E which is in some sense the support of the distribution function F when it is regarded as a measure. Thus, if F is a step function, E consists of the points of discontinuity of F ; if $F(a) = a$ then $E = [0,1]$; if F is continuous and purely singular $\mu(E) = 0$ but $\mu(F(E)) = 1$ (μ is Lebesgue measure). We show in Section IV that the Hausdorff dimension of the set E is equal to H . This extends a theorem proved by Kinney [7] for ergodic Markov chains.

SECTION II

SOME INFORMATION THEORETIC PRELIMINARIES

We review briefly some pertinent facts from Information Theory. It will be convenient to introduce the symbol $P(a_1, \dots, a_n)$ as a shorthand notation for $P(\omega: X_1(\omega) = a_1, \dots, X_n(\omega) = a_n)$ where each $a_i \in \{0, 1, \dots, D-1\}$ for $i = 1, \dots, n$. Also, as noted earlier, all logarithms will be taken to the base D .

To the process $[P, X_n: n \geq 1]$ we associate a set of non-negative numbers.

$$H_n = - \sum_{a_1, \dots, a_n}^* P(a_1, \dots, a_n) \log P(a_1, \dots, a_n) .$$

We mean \sum_{a_1, \dots, a_n}^* to signify that the summation is carried out only over those n -tuples which have a positive probability of occurrence; the number of summands is clearly $\leq D^n$. The numbers H_n which are termed (n -fold) entropies satisfy the following simple inequalities:

$$(a) \quad H_1 \leq \log D = 1$$

$$H_1 = 1 \text{ if and only if } P(\omega: X_1(\omega) = j) = 1/D \text{ for } j = 0, \dots, D-1$$

$$(b) \quad H_1 \geq H_n - H_{n-1} \geq H_{n+1} - H_n \quad n = 2, 3, \dots$$

$$H_1 = H_n - H_{n-1} \text{ for all } n \geq 2 \text{ if and only if the random variables } X_1, X_2, \dots \text{ are independent.}$$

$$(c) \quad H_{n+1} - H_n \geq 0$$

$$H_{n+1} - H_n = 0 \text{ if and only if } P(a_1, \dots, a_{n+1}) \neq 0 \text{ implies that } P(a_1, \dots, a_{n+1}) = P(a_1, \dots, a_n)$$

These inequalities may be derived by judicious application of the following inequality which is valid for $x \geq 0$: $\log x \leq \frac{x-1}{\ln D}$;
 $\log x = \frac{x-1}{\ln D}$ if and only if $x = 1$ where $\ln = \log_e$. For purposes of illustration we sketch the proof of (b).

$$\begin{aligned}
H_{n+1} + H_{n-1} - 2 H_n &= \sum_{a_1, \dots, a_{n+1}}^* P(a_1, \dots, a_{n+1}) \log \frac{P(a_1, \dots, a_n) P(a_2, \dots, a_{n+1})}{P(a_1, \dots, a_{n+1}) P(a_2, \dots, a_n)} \\
&\leq \frac{1}{\ln D} \sum_{a_1, \dots, a_{n+1}}^* P(a_1, \dots, a_{n+1}) \left[\frac{P(a_1, \dots, a_n) P(a_2, \dots, a_{n+1})}{P(a_1, \dots, a_{n+1}) P(a_2, \dots, a_n)} - 1 \right] \\
&= \frac{1}{\ln D} \left[\sum_{a_2, \dots, a_{n+1}}^* \frac{P(a_2, \dots, a_{n+1})}{P(a_2, \dots, a_n)} \sum_{a_1}^* P(a_1, \dots, a_n) - 1 \right] \\
&= \frac{1}{\ln D} \left[\sum_{a_2, \dots, a_{n+1}}^* P(a_2, \dots, a_{n+1}) - 1 \right] = 0
\end{aligned}$$

In order for equality to hold throughout it is necessary and sufficient that $P(a_1, \dots, a_{n+1}) \neq 0$ imply

$$P(a_1, \dots, a_{n+1}) P(a_2, \dots, a_n) = P(a_1, \dots, a_n) P(a_2, \dots, a_{n+1}) \quad (n \geq 1) .$$

This is, however, a necessary and sufficient condition for the random variables X_1, X_2, \dots to be independent.

From (a), (b) and (c) we deduce that the sequence $\{(H_n - H_{n-1})\}$ is monotone non-increasing, and bounded between 0 and 1. Thus the sequence has a limit which we designate by H:

$$H = \lim_{n \rightarrow \infty} (H_n - H_{n-1})$$

Since

$$\lim_{n \rightarrow \infty} (H_n - H_{n-1}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=2}^n (H_j - H_{j-1})$$

it follows readily that

$$H = \lim_{n \rightarrow \infty} \frac{H_n}{n}$$

Accordingly we term H the entropy rate of the process.

- Remarks: 1) The results of this section are valid for sequences which are stationary but not ergodic.
- 2) Observe that $0 \leq H \leq 1$; $H = 1$ if and only if the random variables X_i are both independent and uniformly distributed.

SECTION III

THE FORM OF THE DISTRIBUTION FUNCTION

It will be convenient to define a shift operator T on the space Ω by the rule $(T\omega)_k = \omega_{k+1}$. For a unilateral stationary sequence of random variables T is a measurable, measure preserving, non-invertible transformation.

Lemma 1

Suppose there exists a point $u \in \Omega$ such that $P(u) = \epsilon > 0$. Then $\epsilon = 1/k$ for some positive integer k , F is a step function with k jumps of height $1/k$ and $H = 0$.

Proof

Designate the characteristic function of the set $\{u\}$, by M_u . Then by the Birkhoff ergodic theorem

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} M_u(T^j u) = P(u) = \epsilon > 0.$$

But $M_u(T^j u) = 0$ unless $T^j u = u$. Thus there must exist some smallest positive j , say $j = k$, such that $T^j u = u$. It follows readily that $\epsilon = 1/k$, that the distinct points in Ω with non-zero probability are $u, Tu, \dots, T^{k-1}u$, and that $P(u) = P(Tu) = \dots = P(T^{k-1}u) = 1/k$.

For $n \geq k$ $H_n = \log k$ therefore $H = \lim_{n \rightarrow \infty} \frac{H_n}{n} = 0$.

Lemma 2

If $P(\omega) = 0$ for every $\omega \in \Omega$ then F is continuous.

Proof

A probability measure is countably additive. Consequently $P(\omega : Z(\omega) \leq a)$ is a right continuous function of a and

$P(\omega : Z(\omega) < a)$ is a left continuous function of a . Thus
 $P(\omega : Z(\omega) \leq a) - P(\omega : Z(\omega) < a) = P(\omega : Z(\omega) = a) = 0$ since
there are at most two points in Ω which can be mapped by Z into a
and each of these has 0 probability.

Remark 3) F will be strictly monotone increasing if and only if

$P(a_1, \dots, a_n) > 0$ for every finite sequence a_1, \dots, a_n
of elements chosen from $\{0, 1, \dots, D-1\}$.

Lemma 3

The random variables X_i are independent and uniformly distributed
if and only if $H = 1$. In this case $F(a) = a$.

Proof

If the random variables X_i are independent and uniformly
distributed then $F(g) = g$ at each D -adic rational point, g . The
 D -adic rational points are dense in the unit interval and, by
Lemma 2, F is continuous. Hence $F(a) = a$ for all $0 \leq a \leq 1$. It
was observed in Remark 2 that $H = 1$ if and only if the X_i are
independent and uniformly distributed.

Theorem 1

The distribution function F is one of the following 3 types:

- (1) a step function with k jumps of height $1/k$ where k is
a positive integer,
- (2) continuous and purely singular,
- (3) $F(a) = a$. $0 \leq a \leq 1$.

F is of type (3) if and only if $H = 1$. If F is type (1) then $H = 0$.
If $H > 0$ F is continuous. (Thus if $0 < H < 1$ then F is type (2).)

Proof

It follows from Lemmas 1 and 2 that either F is type (1), in which case $H = 0$, or F is continuous. If F is continuous and $H = 1$ then, by Lemma 3, $F(a) = a$. It remains to show that if F is continuous and $0 \leq H < 1$, then F is purely singular. It suffices to exhibit a set E for which $\mu(E) = 0$ and $\mu(F(E)) = 1$.

Given any $x \in [0,1]$ and any $i \geq 1$ we choose the unique pair of non-negative numbers $\zeta_i = \zeta_i(x)$ and $\eta_i = \eta_i(x)$ such that $\zeta_i(x) + \eta_i(x) = D^{-i}$ and $D^i(x - \eta_i(x))$ and $D^i(x + \zeta_i(x))$ are both non-negative integers.

Consider the set

$$E = \left\{ x : \lim_{i \rightarrow \infty} \frac{\log[F(x + \zeta_i) - F(x - \eta_i)]}{\log[\zeta_i + \eta_i]} = H \right\}$$

and note that for each $x \in E$ and each $\epsilon > 0$ there exists a number $N(x, \epsilon)$ such that if $i > N(x, \epsilon)$ then

$$D^{-i(H+\epsilon)} \leq F(x + \zeta_i) - F(x - \eta_i) \leq D^{-i(H-\epsilon)}$$

If $H < 1$ we can choose $\epsilon > 0$ so that $H + \epsilon < 1$.

Clearly then for each $x \in E$ and $i > N$

$$\frac{F(x + \zeta_i) - F(x - \eta_i)}{\zeta_i + \eta_i} \geq D^{i(1-H-\epsilon)} \rightarrow \infty$$

Since F is a bounded monotone increasing function it must have a finite derivative almost everywhere in the sense of Lebesgue.

(See McShane [9], Thm 34.2, p. 202.) Thus $\mu(E) = 0$.

For $u \in \Omega$ set $[u]_n = \{\omega \in \Omega : \omega_1 = u_1, \dots, \omega_n = u_n\}$

and let $\Omega^* = \{\omega : \lim_{n \rightarrow \infty} \frac{-\log P([\omega]_n)}{n} = H\}$.

By the strong form of McMillan's theorem (Breiman [3] and [4])

$P(\Omega^*) = 1$. The identity

$$F(Z(\omega) + \zeta_i) - F(Z(\omega) - \eta_i) = P([\omega]_i)$$

implies that if $\omega \in \Omega^*$ then $Z(\omega) \in E$. Thus

$$\mu(F(E)) = P(\omega : Z(\omega) \in E) \geq P(\Omega^*) = 1 \quad .$$

McMillan's theorem also implies that if $H > 0$ then $P(\omega) = 0$ for every $\omega \in \Omega$. In this case then, by Lemma 2, F must be continuous.

SECTION IV
ON THE HAUSDORFF DIMENSION OF E

In this section we show that the Hausdorff dimension of the set E constructed in Theorem 1 is equal to H. This is in fact an immediate consequence of a general theorem due to Billingsley [2. Thm 2.4]. We shall give a direct proof, however, establishing in the process Holder conditions on F. The techniques used are similar to those of Kinney [7] and Kinney and Pitcher [8].

Lemma 4

If $x \in E$ then for any $\alpha > 0$

$$\lim_{h \rightarrow 0} [F(x+h) - F(x-h)] (2h)^{-(H+\alpha)} = \infty$$

Proof

The proof follows by observing that we can choose a positive $\delta < \alpha$ and i so large that

$$\begin{aligned} & [F(x + \zeta_i + \eta_i) - F(x - \zeta_i - \eta_i)] [2(\zeta_i + \eta_i)]^{-(H+\alpha)} \\ & \geq 2^{-(H+\alpha)} [F(x + \zeta_i) - F(x - \eta_i)] [\zeta_i + \eta_i]^{-(H+\alpha)} \\ & \geq 2^{-(H+\alpha)} D^{i(\alpha-\delta)} \rightarrow \infty \end{aligned}$$

Lemma 5

For every $\alpha > 0$ there exists a set $A \subseteq E$ such that $\mu(F(A)) = 1$ and for all $x \in A$

$$\lim_{h \rightarrow 0} [F(x+h) - F(x-h)] (2h)^{-H+\alpha} = 0$$

Proof

It suffices given arbitrary $\epsilon > 0$ to exhibit a set A with $\mu(F(A)) > 1 - \epsilon$ whose elements satisfy a Holder condition of the

above type. Choose a positive $\delta < \alpha$ and consider the set of points, B , for which

$$F(x + \zeta_i) - F(x - \eta_i) > (\zeta_i + \eta_i)^{H-\delta}$$

for infinitely many i . Cover B with a countable collection of intervals, C_λ , each of length $< 3\lambda$ and so chosen that for each $x \in B$ the interval

$$[x - 2\eta_i - \zeta_i, x + \eta_i + 2\zeta_i] \subset C_\lambda$$

where i is the smallest integer for which both $\zeta_i + \eta_i < \lambda$ and

$$F(x + \zeta_i) - F(x - \eta_i) > (\zeta_i + \eta_i)^{H-\delta}.$$

Since $\mu(F(B)) = 0$ we can choose λ so small that $\mu(F(C_\lambda)) < \epsilon$.

Set $A = E - C_\lambda$. Clearly $\mu(F(A)) > 1 - \epsilon$. For each $x \in A$ we can choose an integer t so large that $\zeta_t + \eta_t < \lambda$ and for all integers $m \geq t$

$$F(x + \zeta_m) - F(x - \eta_m) < (\zeta_m + \eta_m)^{H-\delta}.$$

This implies that $B \cap [x - \zeta_t - 2\eta_t, x + 2\zeta_t + \eta_t] = \emptyset$ and

further that there exists an integer $g \geq t$ such that for all $s \geq g$

$$F(x + 2\zeta_s + \eta_s) - F(x - \zeta_s - 2\eta_s) \leq 3(\zeta_s + \eta_s)^{H-\delta}.$$

The desired Holder condition for such points x follows easily since $\delta < \alpha$.

Remark 4) Let $\{\alpha_j\}$ be a sequence of positive numbers which decrease monotonically to zero. Let A_j be the set of $x \in E$ for which $F(x+h) - F(x-h) < (2h)^{H-\alpha_j}$ for all sufficiently small positive h . By Lemma 5, $\mu(F(A_j)) = 1$. Clearly the sequence of sets A_j decrease monotonically to a set A with $\mu(F(A)) = 1$ and if $x \in A$ then $F(x+h) - F(x-h) < (2h)^{H-\gamma}$ for any $\gamma > 0$ and all sufficiently small $h > 0$.

We now recall some definitions. The γ dimensional Hausdorff measure of a set $A \subset [0,1]$ is defined by $\Gamma(\gamma; A) = \liminf_{\delta \rightarrow 0} \sum_j \mu(I_j)^\gamma$ where the liminf is taken over all countable sets of intervals $\{I_j\}$ such that $\bigcup_j I_j \supset A$ and $\mu(I_j) < \delta$. The Hausdorff dimension of the set A , $\beta(A)$, is the number with the property that for any $\epsilon > 0$

$$\Gamma[\beta(A) - \epsilon, A] = \infty \quad \Gamma[\beta(A) + \epsilon, A] = 0.$$

We now prove

Theorem 2

If $E^* \subset E$ and $\mu(F(E^*)) > 0$ then $\beta(E^*) = H$.

Proof

Given any fixed $\epsilon > 0$ there exists for each $x \in E^*$ a smallest $i \geq n$ such that

$$F(x + \zeta_i) - F(x - \eta_i) > [\zeta_i + \eta_i]^{H+\epsilon}.$$

We can thus construct a countable set of mutually disjoint intervals $\{I_i^{(n)}\}$ $i = 1, 2, \dots$ each of length $\leq D^{-n}$ and so chosen that

$$\bigcup_i I_i^{(n)} \supset E^* \text{ and } \mu(F(I_i^{(n)})) \geq \mu(I_i)^{H+\epsilon}.$$

Thus

$$1 \geq \mu(F(\bigcup_i I_i^{(n)})) \geq \sum_i (\mu(I_i^{(n)}))^{H+\epsilon}.$$

Hence

$$\Gamma(H+\epsilon, E^*) \leq 1 \quad \text{and} \quad \beta(E^*) \leq H.$$

To prove the inequality in the other direction fix $\epsilon > 0$ and choose $h_0 > 0$ and a set $A(h_0) \subset E^*$ so that $\mu(F(A(h_0))) \geq \delta > 0$

and for each $x \in A(h_0)$ and every $h \leq h_0$ $F(x+h) - F(x-h) \leq (2h)^{H-\epsilon}$.

Now cover $A(h_0)$ with a set of intervals I_i , each chosen with length $< h_0$. We can assume that each interval I_i contains at least one point, $x_i \in A(h_0)$. Let J_i be the smallest interval symmetric about x_i which contains I_i . Then $\mu(J_i) \leq 2\mu(I_i)$ and correspondingly

$$\sum (2\mu(I_i))^{H-\epsilon} \geq \sum (\mu(J_i))^{H-\epsilon} \geq \sum \mu(F(J_i)) \geq \mu(f(A(h_0))) \geq \delta.$$

Since the chosen covering of $A(h_0)$ was arbitrary, aside from the fact that the length of each interval was constrained to be smaller than h_0 , $\Gamma(H-\epsilon, A(h_0)) \geq \delta$. Hence $\beta(A(h_0)) \geq H$.

But $E^* \supset A(h_0)$ thus $\beta(E^*) \geq H$.

Remark 5) The proof as presented is valid for $0 \leq H \leq 1$. The effort expended is, however, only really needed for distribution functions of the second type. For if F is type (1) the set E consists of the k jump points and if F is type (3) $E = [0, 1]$. The Hausdorff dimension of a finite or even Countable set of points is 0; whereas the Hausdorff dimension of a bounded set of positive Lebesgue measure is 1.

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